A taxonomy of Dedekind finite and reflexive rings

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Outline

Ring Types

1. **zero-divisorless(ZD)**: for $a, b \in R$,
   $ab = 0 \Rightarrow a = 0 \text{ or } b = 0$.
2. **prime**: for $I, J \lhd R$,
   $IJ = 0 \Rightarrow I = 0 \text{ or } J = 0$.
3. **reduced**: for $a \in R$,
   $a^2 = 0 \Rightarrow a = 0$.
4. **semiprime**: for $I \lhd R$,
   $I^2 = 0 \Rightarrow I = 0$.
5. **symmetric**: for $a, b, c \in R$,
   $abc = 0 \Rightarrow bac = 0$.
6. **symflexive**: for $I, J, K \lhd R$,
   $IJK = 0 \Rightarrow JIK = 0$.
7. **reversible**: for $a, b \in R$,
   $ab = 0 \Rightarrow ba = 0$.
8. **reflexive**: for $I, J \lhd R$,
   $IJ = 0 \Rightarrow JI = 0$.

Ring Types

All of these ring types except for symflexive rings have been well studied.

Symflexive rings were originally introduced under the name *ideal-symmetric*.

In the same way, prime rings could be called ideal-ZD, semiprime rings called ideal-reduced, and reflexive rings called ideal-reversible.

Of course, ZD rings are domains and sometimes referred to as completely prime rings.

Reduced rings are sometimes referred to as completely semiprime.

In the same way then, reversible rings are completely reflexive and symmetric rings are completely symflexive.

Ring Class Containments

$$
\begin{array}{ccc}
\text{ZD} & & \text{prime} \\
(\textit{completely prime}) & \Longrightarrow & (\textit{ideal-ZD}) \\
\Downarrow & & \Downarrow \\
\text{reduced} & & \text{semiprime} \\
(\textit{completely semiprime}) & \Longrightarrow & (\textit{ideal-reduced}) \\
\Downarrow & & \Downarrow \\
\text{symmetric} & & \textbf{symflexive} \\
(\textit{completely symmetric}) & \Longrightarrow & (\textit{ideal-symmetric}) \\
\Downarrow & & \Downarrow \\
\text{reversible} & & \text{reflexive} \\
(\textit{completely reflexive}) & \Longrightarrow & (\textit{ideal-reversible})
\end{array}
$$

Constructing this diagram was how we rediscovered symflexive rings.

What ring property can be added to the ideal version of a property to guarantee the element version of the property?

More Ring Types

For a ring R, ...

- $U(R)$ is the set of invertible elements of R,
- $N(R)$ is the set of nilpotent elements of R,
- an element $a \in R$ is **fine** if $a \in U(R) + N(R)$.
- $\Phi(R)$ is the set of fine elements of R,
- $L \lhd R$ is **nil** if every element of L is nilpotent.
- $L \lhd R$ is **locally nilpotent** if, for any finite set $L' \subset L$ there exists an integer $n > 0$, such that any product of n elements from L' is 0.
- $L \lhd R$ is **prime** if for $I, J \lhd R$, $IJ \subset L \Rightarrow I \subset L \text{ or } J \subset L$,
- $\mathrm{Nil}^*(R)$ is the sum of all nil ideals,
- $L(R)$ is the largest locally nilpotent ideal,
- $\mathrm{Nil}_*(R)$ is the intersection of all prime ideals.
- $\mathrm{Nil}_*(R) \subset L(R) \subset \mathrm{Nil}^*(R) \subset N(R)$

More Ring Types

1. **division**: $R \setminus 0 = U(R)$,
2. **fine**: $R \setminus 0 = \Phi(R)$,
3. **simple**: R has no non-trivial 2-sided ideals.
4. **left(right) primitive**: R has a faithful simple left(right) R-module.
5. **dedekind finite(DF)**: for all $a, b \in R$, $ab = 1$ implies $ba = 1$.
6. **NR**: $N(R)$ is a subring of R, equivalently, $N(R)$ is an additive subgroup of R.
7. **NI**: $N(R)$ is an ideal of R, equivalently, $N(R) = \mathrm{Nil}^*(R)$.
8. **weakly 2-primal**: $N(R) = L(R)$
9. **2-primal**: $N(R) = \mathrm{Nil}_*(R)$.
10. **PS I**: for every $a \in R$, $R/\mathrm{ann}_r^R(aR)$ is 2-primal.
11. **semicommutative**: for all $a, b \in R$, $ab = 0 \Rightarrow aRb = 0$,

More Ring Types

Clearly, division rings are fine.

Lam showed fine rings are simple.

Simple rings are left primitive and left primitive rings are prime.

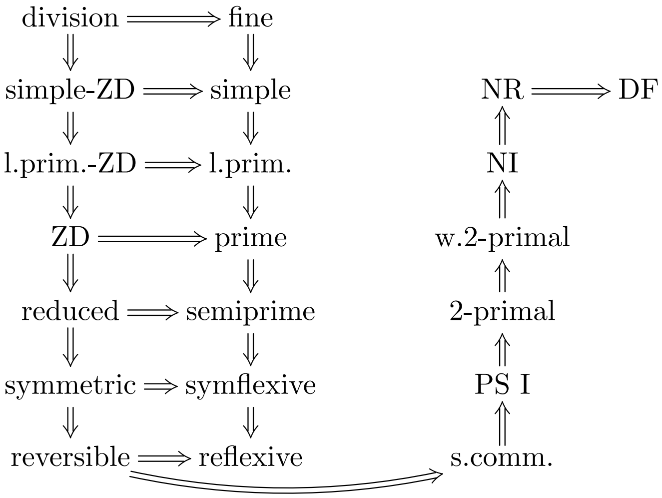
Reversible rings are semicommutative.

Semicommutative rings are PS I.

The following chain follows from the definitions.

$$\text{PS I} \Rightarrow \text{2-primal} \Rightarrow \text{w.2-primal} \Rightarrow \text{NI} \Rightarrow \text{NR}$$

Ring Class Containments



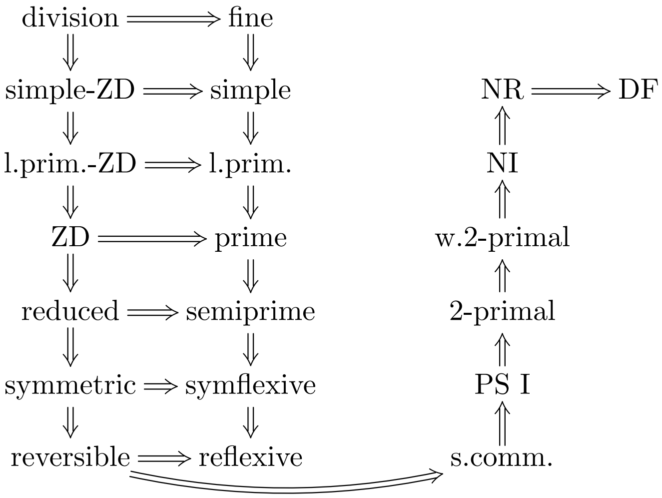
From Ideals to Elements

What ring property can be added to the ideal version of a property to guarantee the element version of the property?

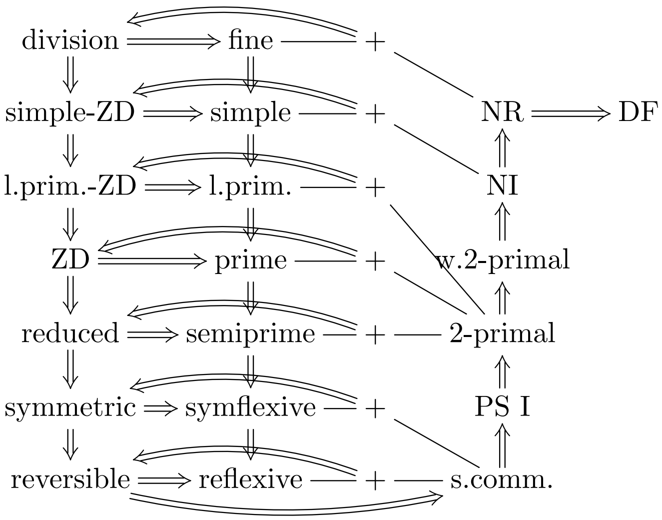
A ring is ...

1. division $\Leftrightarrow$ fine and NR
2. simple-ZD $\Leftrightarrow$ simple and NI
3. left primitive-ZD $\Leftrightarrow$ left primitive and 2-primal
4. ZD $\Leftrightarrow$ prime and 2-primal
5. reduced $\Leftrightarrow$ semiprime and 2-primal
6. symmetric $\Leftrightarrow$ symflexive and semicommutative
7. reversible $\Leftrightarrow$ reflexive and semicommutative

Ring Class Containments



Ring Class Containments



Differentiating Examples

$$\frac{\mathbb{F}_4[v;\sigma][u]}{\langle u^2 + v^2, uv \rangle}$$

is a finite symflexive nonsemicommutative abelian ring.

Finite abelian rings are PS I.

Since this is nonsemicommutative, it is nonreversible and nonsymmetric.

So, semicommutativity cannot be weakened to PS I in the characterizations of reversible nor symmetric rings.

Differentiating Examples

An example, due originally to J. Ram, is a prime weakly-2-primal non-2-primal ring.Let K be a field and in the polynomial ring $K[\{t_i\}_{i\in\mathbb{Z}}]$ define the ideal $I = (\{t_{i_1}t_{i_2}t_{i_3} \mid i_3 - i_2 = i_2 - i_1 > 0\})$. Let

$$A = K[\{t_i\}_{i\in\mathbb{Z}}]/I.$$

Then A is a commutative K-algebra with generators t_i ($i \in \mathbb{Z}$) subject to the relations $t_{i_1}t_{i_2}t_{i_3} = 0$, where $i_1 < i_2 < i_3$ range over all increasing arithmetic progressions of length 3 in $\mathbb{Z}$. Let σ be the K-automorphism of A defined by $\sigma(t_i) = t_{i+1}$ for all $i \in \mathbb{Z}$. Lam shows

$$R = A[x, \sigma]$$

is prime with non-trivial Levitzki radical. Since the lower nilradical is trivial for prime rings and $L(R) \neq 0$, the ring R is non-2-primal. Marks shows $L(R) = N(R)$, i.e R is weakly-2-primal ring. So, 2-primal cannot be weakened to weakly-2-primal in the characterization of reduced nor ZD rings.

Differentiating Examples

Chebotar et. al. give an example of a simple NR ring which is not ZD. Hence, the NI condition cannot be weakened to NR in the characterization of simple-ZD rings.

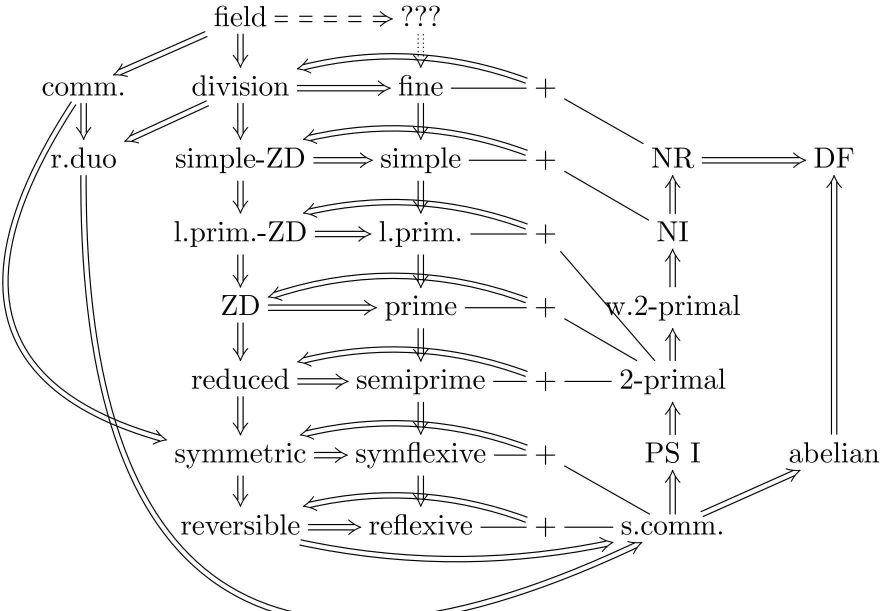
Simple artinian rings are fine rings. A simple artinian ring is of the form $M_n(D)$ for some division ring D.

Such a ring is non-NR ($e_{12} + e_{21}$ is not nilpotent).

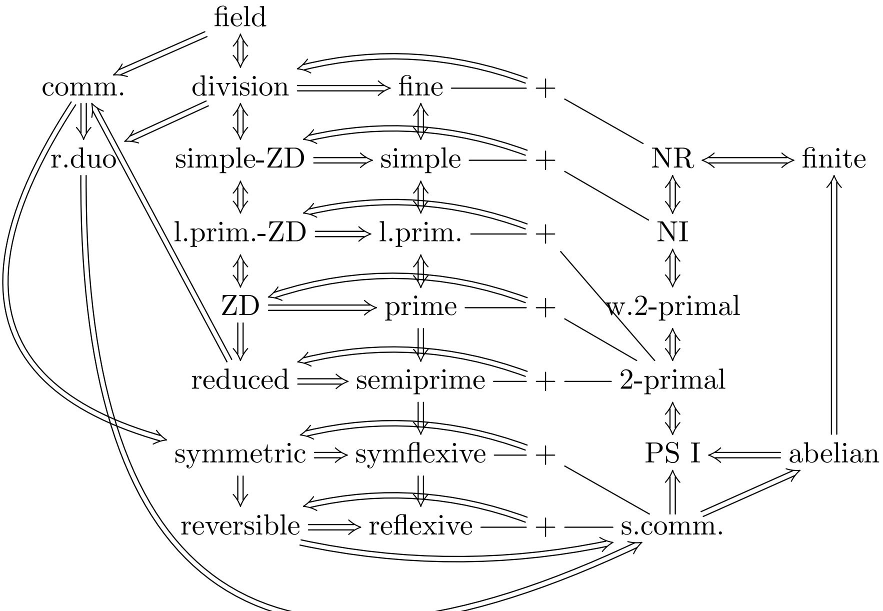
Since $M_2(\mathbb{F}_2)$ is a fine DF ring that is not a division ring, the NR condition cannot be weakened to DF in the characterization of division rings.

Question: Is a left primitive weakly 2-primal ring, ZD?

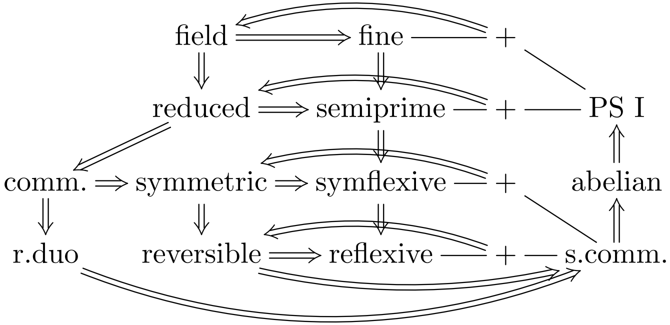
Ring Class Containments



Finite Ring Class Containments



Finite Ring Class Containments



Minimal Rings

$\mathbb{F}_2 Q_8$ is a minimal reversible nonsymmetric ring.

$\mathbb{F}_2 D_8$ is a minimal abelian reflexive nonsemicommutative ring. This is also nonsymflexive.

$$\frac{\mathbb{F}_4[v;\sigma][u]}{\langle u^2+v^2, uv\rangle}$$

is a minimal abelian symflexive nonsemicommutative ring.

$$R = \begin{pmatrix} \frac{\mathbb{F}_2[x]}{\langle x^2\rangle} & \mathbb{F}_2 \\ \mathbb{F}_2 & \frac{\mathbb{F}_2[x]}{\langle x^2\rangle} \end{pmatrix}$$

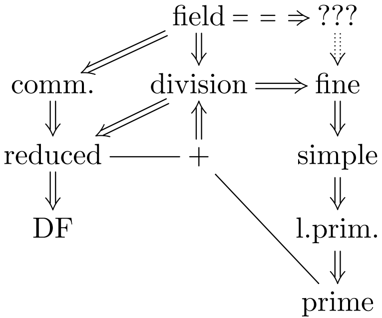
is a minimal reflexive nonsymflexive PS I nonabelian ring.

There does not exist a finite symflexive nonsemiprime PS I nonabelian ring. Does an infinite example exist?

Regular Ring Class Containments

Let R be a von Neumann regular ring. Then R is semiprime and R is a division ring if and only if R is a domain. Furthermore, the following are equivalent.

1. R is reduced
2. R is right duo
3. R is abelian
4. R is NR



Questions

- We recently identified a few NI non-weakly-2-primal rings. Others?
- Is a left primitive weakly 2-primal ring a domain?
- What are the various minimal (w.r.t. order) class differentiating examples?
- How does the diagram collapse under various finiteness conditions (artinian, noetherian, ...)? We have done it for finite and von Neumann regular.
- For which combinations of ring properties A and B are deeper studies of A non-B rings warranted?
- In the characterizations of Morita context rings for simple, prime, semiprime, symflexive, and reflexive it can be seen that corner rings of such rings must be of the same type. Is this also true for fine rings (posed by Lam in his fine rings paper)?

Related Work

Henry Chimal-Dzul and Steve Szabo.
Minimal reflexive nonsemicommutative rings.
J. Algebra Appl., 2020.

Hunry Chimal-Dzul, Sergio López-Permouth, and Steve Szabo.
Characterizations of some classes of morita context rings.
Comm. Algebra, 2021.

Hunry Chimal-Dzul and Steve Szabo.
Symflexive rings.
in preparation.

Hunry Chimal-Dzul and Steve Szabo.
Classification of nonsemicommutative rings.
to appear in Acta Mathematica Hungarica, 2021.

Steve Szabo.
Minimal reversible nonsymmetric rings.
J. Pure Appl. Algebra, 223(11):4583–4591, 2019.

Steve Szabo.
Some minimal rings related to 2-primal rings.
Comm. Algebra, 47(3):1287–1298, 2019.

Thank You


I DID IT! I DID IT!
SOMEHOW I IMAGINED THIS EXPERIENCE WOULD BE MORE REWARDING.
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3·7